TOTAL \( k \)-DISTANCE DOMINATION CRITICAL GRAPHS

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Abstract. A set \( S \) of vertices in a graph \( G = (V, E) \) is called a total \( k \)-distance dominating set if every vertex in \( V \) is within distance \( k \) of a vertex in \( S \). A graph \( G \) is total \( k \)-distance domination-critical if \( \gamma_k^t(G - x) < \gamma_k^t(G) \) for any vertex \( x \in V(G) \). In this paper, we investigate some results on total \( k \)-distance domination-critical of graphs.

1. Introduction

The terminology and notation in \[3\] will be used throughout. The distance \( d_G(u, v) \) between two vertices \( u \) and \( v \) of \( G \) is the length of the shortest \( u-v \) path if such path exists, otherwise \( d_G(u, v) = \infty \). The open \( k \)-neighborhood \( N_k(X) \) of a subset \( X \subseteq V(G) \) is the set of vertices in \( V(G) - X \) of distance at most \( k \) from each element of \( X \) and the closed \( k \)-neighborhood is defined by \( N_k[X] = N_k(X) \cup X \). If \( X = \{v\} \) is a single vertex, then we denote the (closed) \( k \)-neighborhood of \( v \) by \( N_k(v) \) (\( N_k[v] \), respectively). The (closed) 1-neighborhood of a vertex \( v \) or a set \( X \) of vertices is usually denoted \( N(v) \) or \( N(X) \), respectively (\( N[v] \) or \( N[X] \), respectively). The minimum \( k \)-degree \( \delta_k(G) \) equals \( \min\{|N_k(v)| : v \in V\} \), while the maximum \( k \)-degree \( \Delta_k(G) \) equals \( \max\{|N_k(v)| : v \in V\} \). For a set \( S \subseteq V(G) \), we denote the subgraph of \( G \) induced by \( S \) by \( (S) \). The \( k \)-th power of a graph \( G \) is the graph \( G^k \) with vertex set \( V(G^k) = V(G) \) and edge set \( E(G^k) = \{xy : 1 \leq d_G(x, y) \leq k\} \).

Given \( k \leq n \), place \( n \) vertices around a circle, equally spaced. If \( k \) is even, form the Harary graph \( H_{k,n} \) by making each vertex adjacent to the nearest \( k/2 \) vertices in each direction around the circle, \[12\]. It is well known that every \( m \)-th power of a cycle \( C_n \) with \( n \) vertices \( C_{mn}^n \) is \( H_{2m,n} \).

The circulant graph \( C(n; M) \) is the graph with the vertex set \( V(C(n; M)) = \{v_i|0 \leq i \leq n - 1\} \) and the edge set \( E(C(n; M)) = \{v_iv_j|0 \leq i \leq n - 1, 0 \leq j \leq n - 1, (i - j)(mod n) \in M\} \), \( M \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\} \), \[11\][12].

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The dominating set (total dominating set) \( D \) of a graph \( G \) is a set of vertices of \( G \) such that every vertex of \( V(G) - D \) (respectively, \( V(G) \)) is adjacent to some vertex of \( D \). The domination number \( \gamma(G) \) (total domination number \( \gamma_t(G) \)) of \( G \) is the minimum cardinality of a dominating set (total dominating set) of \( G \).

A subset \( S \subseteq V(G) \) is a \( k \)-distance dominating set if every vertex in \( V - S \) is within distance \( k \) of at least one vertex in \( S \) for integers \( k > 1 \). That is \( N_k[S] = V(G) \). A subset \( S \subseteq V(G) \) is a total \( k \)-distance dominating set (TkDDS for short) if every vertex \( u \in V(G) \) is within distance \( k \) from at least one vertex in \( S \) other than itself. The minimum cardinality of a (total) distance \( k \)-dominating set in \( G \) is the (total) distance \( k \)-domination number of \( G \), denoted by \( \gamma^k(G) \) (\( \gamma_t^k(G) \), respectively). Any TkDDS of cardinality \( \gamma_t^k(G) \) is called a \( \gamma_t^k \)-set of \( G \), \cite{4}.

2. Preliminary results

By the definitions of TkDDS and \( G^k \) it immediately follows:

**Observation 2.1.** Let \( G \) be a nontrivial connected graph. Then a set \( D \subseteq V(G) \) is a TkDDS of \( G \) if and only if \( D \) is a total dominating set of \( G^k \).

**Corollary 2.2.** Each \( \gamma_t \)-set of \( G^k \) is a \( \gamma_t^k \)-set of \( G \) and vice versa, that is, \( \gamma_t^k(G) = \gamma_t(G^k) \).

An end-vertex is a vertex of degree one and a support vertex is one that is adjacent to an end-vertex. Let \( S(G) \) be the set of support vertices of \( G \). We say that a vertex \( v \in V(G) - S(G) \) is a \( \gamma_t^k \)-critical if \( \gamma_t^k(G - v) < \gamma_t^k(G) \).

**Observation 2.3.** Let \( v \) be a \( \gamma_t^k \)-critical vertex of a graph \( G \). Then:

(i) Each vertex of \( N_k(v) \) is not in any \( \gamma_t^k \)-set of \( G - v \);

(ii) if \( T \) is a \( \gamma_t^k \)-set of \( G - v \), then \( T \cup \{u\} \) is a \( \gamma_t^k \)-set of \( G \) for every \( u \in N_k(v) \);

(iii) \( \gamma_t^k(G) = \gamma_t^k(G - v) + 1 \).

**Proof.** (i) If each vertex of \( N_k(v) \) is in any \( \gamma_t^k \)-set of \( G - v \), then \( \gamma_t^k(G - v) \geq \gamma_t^k(G) \), a contradiction.

(ii) and (iii) If \( T \) is a \( \gamma_t^k \)-set of \( G - v \) and \( u \in N_k(v) \), then \( T \cup \{u\} \) is a TkDDS of \( G \) and \( |T \cup \{u\}| = \gamma_t^k(G - v) + 1 \leq \gamma_t^k(G) \). Thus \( \gamma_t^k(G) = \gamma_t^k(G - v) + 1 \) and \( T \cup \{u\} \) is a \( \gamma_t^k \)-set of \( G \). \( \square \)

Since total domination may not be defined for a graph with isolated vertices, we say that a graph \( G \) is total \( k \)-distance domination vertex critical, or just \( \gamma_t^k \)-critical, if every vertex of \( V(G) - S(G) \) is a \( \gamma_t^k \)-critical vertex. If \( G \) is \( \gamma_t^k \)-critical, and \( \gamma_t^k(G) = r \), then we say that \( G \) is \( r \)-\( \gamma_t^k \)-critical. Note that a graph is vertex \( \gamma_t^k \)-critical if and only if each its component is \( \gamma_t^k \)-critical.

**Corollary 2.4.** If \( G \) is a connected \( \gamma_t^k \)-critical graph, then \( \gamma_t^k(G - v) = \gamma_t^k(G) - 1 \) for any \( v \in V(G) - S(G) \).

**Proposition 2.5.** Let \( G \) be a graph.
Proof. The case of \( k = 1 \) is trivial, so we assume that \( k \geq 2 \).

(i) Since \( v \) is \( \gamma^k_t \)-critical, \( \gamma^k_t(G-v) = \gamma^k_t(G) - 1 \). By Corollary 2.2, \( \gamma^k_t(G-v) = \gamma_t((G-v)^k) \) and \( \gamma^k_t(G) = \gamma_t(G^k) \). Since the total domination number does not increase when edges are added to a graph and since \( (G-v)^k \) is a spanning subgraph of \( G^k-v \), it follows that \( \gamma_t((G-v)^k) \geq \gamma_t(G^k-v) \). Thus \( \gamma_t(G^k-v) \leq \gamma^k_t(G-v) = \gamma^k_t(G) - 1 = \gamma_t(G^k) - 1 \).

(ii) Since \( v \) is a \( \gamma_t \)-critical vertex of \( G^k \), \( \gamma_t(G^k-v) = \gamma_t(G^k) - 1 = \gamma^k_t(G) - 1 \) and no neighbor of \( v \) in \( G^k \) belongs to some \( \gamma_t \)-set of \( G^k-v \). Hence each \( \gamma_t \)-set of \( G^k-v \) is a total dominating set of \( (G-v)^k \) which implies \( \gamma_t(G^k-v) \geq \gamma_t((G-v)^k) \). Since always \( \gamma_t(G^k-v) \leq \gamma_t((G-v)^k) \), the equality \( \gamma_t(G^k-v) = \gamma_t((G-v)^k) \) holds. Thus \( \gamma^k_t(G) - 1 = \gamma^k_t(G-v) \) as required. \( \square \)

**Corollary 2.6.** Let \( G \) be a graph with \( \delta(G) \geq 2 \). Then \( G \) is \( \gamma^k_t \)-critical if and only if \( G^k \) is \( \gamma_t \)-critical.

3. Total \( k \)-distance domination

We start this section with an important result from [2].

**Theorem 3.1.** [2] Let \( G \) be a \( \gamma_t \)-critical graph of order \( n \). Then \( n \leq \Delta(G)(\gamma_t(G) - 1) + 1 \).

In what follows, for any vertex \( v \) in \( G \), \( S_v \) denotes a total \( k \)-distance dominating set of the subgraph \( G_v = G - v \) with minimum size, and \( S_v^u \) denotes the set \( S_v \cup \{u\} \) for \( u \in V(G) \).

**Theorem 3.2.** Let \( G \) be a \( \gamma^k_t \)-critical graph of order \( n \). Then \( n \leq \Delta_k(G)(\gamma^k_t(G) - 1) + 1 \).

**Proof.** Let \( v \in V(G) - S(G) \). Total criticality of \( G \) implies that there exists a \( S_v \) with \( |S_v| = \gamma^k_t(G) - 1 \) for \( G_v \). Since each vertex of \( S_v \) can \( k \)-distance dominate at most \( \Delta_k(G) \) vertices, \( S_v \) can \( k \)-distance dominate at most \( \Delta_k(G)(\gamma^k_t(G) - 1) \) vertices, which implies that \( n = |V(G_v)| + 1 \leq \Delta_k(G)(\gamma^k_t(G) - 1) + 1 \). \( \square \)

In [7] it has been given the result:

**Theorem 3.3.** ([7], Theorem 1) Any \( \gamma_t \)-critical graph \( G \) of order \( n = \Delta(G)(\gamma_t(G) - 1) + 1 \) is regular.

One can have the following which shows that Theorem 3.3 cannot be generalized for total \( k \)-distance domination of \( G \).

**Theorem 3.4.** There is no \( \gamma^k_t \)-critical graph of order \( n = \Delta_k(G)(\gamma^k_t(G) - 1) + 1 \).

**Proof.** Let \( G \) be a \( k - \gamma^k_t \)-critical graph of order \( \Delta_k(G)(\gamma^k_t(G) - 1) + 1 \). So \( G^k \) is a \( k - \gamma_t \)-critical graph of order \( \Delta_{G^k}(k-1) + 1 \). Let \( v \in G^k \) and \( S_v \) is a \( \gamma_t(G^k-v) \)-set. Then \( S_v \) is an efficient total dominating set for \( G^k-v \) and no neighbor of \( v \) is in \( S_v \). Hence in \( G^k \) each edge belongs to triangles, there is at least one vertex \( u \in G^k \) such that \( |N_{G^k}(u) \cap S_v| \neq 1 \). Therefore by above assumptions, the bound \( \Delta(G^k)(\gamma_t(G^k) - 1) + 1 \) is not attainable. \( \square \)
Combining of Theorems 3.2 and 3.4 we have the following corollary:

**Corollary 3.5.** If $G$ is a $\gamma_t^k$-critical graph of order $n$, then $n \leq \Delta_k(G)(\gamma_t^k(G) - 1)$.

We need the results from [10] and [2].

**Lemma 3.6.** [10] For each $k \geq 1$, if the vertices $x$ and $y$ are two vertices in $G$ such that $\rho_G(x, y) = d(G)$, then $d_{G_k}(x, y) = d(G^k)$. Furthermore, $d(G^k) = \lceil \frac{d(G)}{k} \rceil$.

**Theorem 3.7.** [2] For $m \leq 8$, the diameter of a $m$-$\gamma_t$-critical graph is at most the value given by the following table:

<table>
<thead>
<tr>
<th>$m$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{\text{iam}}$</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>11</td>
</tr>
</tbody>
</table>

A generalization of Lemma 3.6 and Theorem 3.7 is the next result.

**Proposition 3.8.** If $G$ is an $m$-$\gamma_t^k$-critical graph for $m \leq 8$, then $d(G) \leq 11k$.

**Proof.** Using Lemma 3.6 and Theorem 3.7, we have $\frac{d(G)}{k} \leq d(G^k) \leq 11$. \(\square\)

**Example 3.9.** For $n \leq 7$, $\gamma_t^2(C_n) = \gamma_t^2(P_n) = 2$, and for $n \geq 8$, $\gamma_t^2(C_n) = \gamma_t^2(P_n) = \lceil \frac{n}{2} \rceil + \lceil \frac{n-1}{7} \rceil - 1$ if $n \equiv 2 \pmod{7}$ and $\gamma_t^2(C_n) = \gamma_t^2(P_n) = \lceil \frac{n}{2} \rceil + \lceil \frac{n-1}{7} \rceil$ otherwise.

**Example 3.10.** $\gamma_t^3(C_n) = \gamma_t^3(P_n) = \lceil \frac{n}{3} \rceil + 1$ if $n = 10k + 4$, and $\gamma_t^3(C_n) = \gamma_t^3(P_n) = \lceil \frac{n}{3} \rceil$ otherwise.

We have the result from [6] that is useful for other results.

**Theorem 3.11.** ([6], Theorem 13) Let $H_{2m,n}$ be a Harary graph with $n$ vertices and $n = (3m+1)l+r$, where $0 \leq r \leq 3m$. Then

$$\gamma_t(H_{2m,n}) = \begin{cases} 2l & \text{if } r = 0 \\ 2l + 1 & \text{if } 1 \leq r \leq m \\ 2l + 2 & \text{if } m + 1 \leq r \leq 3m. \end{cases}$$

Now using Theorem 3.11 to follow the following result.

**Theorem 3.12.** $H_{2k,n}$ is $\gamma_t$-critical if and only if $3k + 1 \mid n - 1$ or $3k + 1 \mid n - (k + 1)$.

**Proof.** Corollary 2.2 and Theorem 3.11 imply that $\gamma_t^k(C_n) = \gamma_t^k(C_n) = \gamma_t(H_{2k,n})$. Now we show that $H_{2k,n}$ is $\gamma_t$-critical if and only if $3k + 1 \mid n - 1$ or $3k + 1 \mid n - (k + 1)$.

It is easy to see that, two adjacent vertices dominate at most $3k+1$ vertices and three adjacent vertices dominate at most $4k+1$ vertices. Let $n = (3k+1)l+r$ where $0 \leq r \leq 3k$. If $r = 0$, then $\gamma_t(H_{2k,n}) = 2l$. Now consider $v$ as any vertex and $H_{2k,n} - v$ then the size of $H_{2k,n} - v$ is $(3k+1)(l-1) + 3k$. This shows $\gamma_t(H_{2k,n} - v) = 2l$, it is not critical, for $n = 3k + 1)l$. 

If \( r = 1 \), then \( \gamma_t(H_{2k,n}) = 2l + 1 \). Now consider \( H_{2k,n} - v_n \) (note that all vertices play same role), then the set

\[
S = \{ v_{k+1}, v_{2k+1}, v_{4k+2}, v_{5k+2}, \ldots, v_{(3l-5)k+l}, v_{(3l-4)k+l}, v_{(3l-2)k+l}, v_{(3l-1)k+l} \}
\]

is a total dominating set of \( H_{2k,n} - v_n \) with size \( |S| = 2l \). Thus \( H_{2k,n} \) for \( n = (3k+1)l + 1 \) is total critical.

Let \( 2 \leq r \leq k \). Then \( \gamma_t(H_{2k,n}) = 2l + 1 \). Since \( 2l \) vertices totally dominate at most \((3k+1)l \) vertices and for \( 2 \leq r \leq k \) the graph \( H_{2k,n} - v \) has at least \((3k+1)l + 1 \) vertices, then \( \gamma_t(H_{2k,n} - v) = 2l + 1 \). Therefore \( H_{2k,n} \) is not total critical, where \( n = (3k+1)l + r \) and \( 2 \leq r \leq k \).

Let \( r = k + 1 \). Then \( \gamma_t(H_{2k,n}) = 2l + 2 \). Same as above consider \( H_{2k,n} - v_n \), then the set vertex

\[
S = \{ v_{k+1}, v_{2k+1}, v_{4k+2}, v_{5k+2}, \ldots, v_{(3l-5)k+l}, v_{(3l-4)k+l}, v_{(3l-2)k+l}, v_{(3l-1)k+l}, v_{(3l)k+l} \}
\]

is a total dominating set of \( H_{2k,n} - v_n \) with size \( |S| = 2l + 1 \). Thus \( H_{2k,n} \) for \( n = (3k+1)l + k + 1 \) is total critical.

Let \( k + 2 \leq r \leq 3k \). Then \( \gamma_t(H_{2k,n}) = 2l + 2 \). Since \( 2l + 1 \) vertices totally dominate at most \((3k+1)l + k \) vertices and for each \( v \), \( H_{2k,n} - v \) has at least \((3k+1)l + k + 1 \) vertices. And so \( \gamma_t(H_{2k,n} - v) = 2l + 2 \). Thus \( H_{2k,n} \) for \( n = (3k+1)l + r \) for \( k + 2 \leq r \leq 3k \) is not total critical. Therefore the proof is complete.

From Theorem 3.12 for \( H_{2k,n} \) where \( 3k + 1 \mid n - 1 \) or \( 3k + 1 \mid n - (k + 1) \) we have \( n < \Delta(G)(\gamma_t(G) - 1) + 1 \). This result shows that the converse of Theorem 3.1(ii) is not true.

As an immediate result from Theorems 3.11 and 3.12 we have:

**Proposition 3.13.** Let \( C_n \) be a cycle with \( n \) vertices and \( n = (3k + 1)l + r \), where \( 0 \leq r \leq 3k \). Then

\[
\gamma_t^k(C_n) = \begin{cases} 
2l & \text{if } r = 0 \\
2l + 1 & \text{if } 1 \leq r \leq k \\
2l + 2 & \text{if } k + 1 \leq r \leq 3k.
\end{cases}
\]

(ii) \( C_n \) is \( \gamma_t^k \)-critical if and only if \( 3k + 1 \mid n - 1 \) or \( 3k + 1 \mid n - (k + 1) \).

**Proof.** By Corollary 2.2 \( \gamma_t^k(C_n) = \gamma_t(C_n^k) \). Since \( C_n^k = H_{2k,n} \), (i) holds by Theorem 3.11 and (ii) holds by Theorem 3.12.

**Observation 3.14.** There is no \( \gamma_t^k \)-critical graph with \( \Delta_k = 2 \).

**Proof.** Let \( G \) be a \( \gamma_t^k \)-critical graph with \( \Delta_k = 2 \) and \( x \in V(G) \). Since \( \Delta_k = 2 \), \( \deg(x) \leq 2 \). Therefore \( G \) is \( P_2, P_3 \) or \( C_3 \) and none of them is \( \gamma_t^k \)-critical graph.

**Proposition 3.15.** Let \( G \) be a corona of \( G' \) with \( \delta(G') \geq 2 \). Then \( G \) is \( \gamma_t \)-critical, and for \( k \geq 2 \), \( G \) is not necessary \( \gamma_t^k \)-critical.
Proof. It is clear that $\gamma_t(G) = |V(G')|$. If $v$ is a vertex in $V(G) \setminus S(G)$, then $\gamma_t(G - v) = |V(G')| - 1$.

For $k \geq 2$, let $G' = K_n$ ($n \geq 3$) and $G$ be corona of $K_n$. Then $\gamma_t^k(G) = \gamma_t^k(G - v)$ for $k \geq 2$. □

**Theorem 3.16.** If $G$ has a cut vertex, and $G$ is not corona of a graph $G'$ with $\delta(G') \geq 2$, then $G$ is not a $\gamma_t^k$-critical graph with $k \geq 1$.

Proof. Let $u$ be a cut vertex of $G$ such that $u \not\in S(G)$, and $G - u$ has two components, say $G_1$ and $G_2$. Suppose on the contrary that $G$ is a $\gamma_t^k$-critical graph and $S$ is a $\gamma_t^k$-set for $G$. Let $S_1$ and $S_2$ be the $\gamma_t^k$-set for $G_1$ and $G_2$. Let $n_1 = |S \cap G_1|$ and $n_2 = |S \cap G_2|$. Since $G$ is a $\gamma_t^k$-critical graph, $\gamma_t^k(G - u) = \gamma_t^k(G_1) + \gamma_t^k(G_2)$, and then we will have $\gamma_t^k(G) = \gamma_t^k(G_1) + \gamma_t^k(G_2) + 1$.

If $u \in S$, one of the followings holds:

(i) $n_1 = \gamma_t^k(G_1)$, $n_2 = \gamma_t^k(G_2)$;
(ii) $n_1 > \gamma_t^k(G_1)$, $n_2 < \gamma_t^k(G_2)$;
(iii) $n_1 < \gamma_t^k(G_1)$, $n_2 > \gamma_t^k(G_2)$.

Remove the vertex $u$ from $\gamma_t^k$-set. It is easy to see that $G$ can be total $k$-distance dominated by $S_1 \cup S_2 \cup \{x\}$, where $x \in N_k(u)$, a contradiction.

Let now that $u \not\in S$. We have one of the followings:

(a): $n_1 = \gamma_t^k(G_1)$, $n_2 > \gamma_t^k(G_2)$;
(b): $n_1 > \gamma_t^k(G_1)$, $n_2 = \gamma_t^k(G_2)$;
(c): $n_1 < \gamma_t^k(G_1)$, $n_2 > \gamma_t^k(G_2)$;
(d): $n_1 > \gamma_t^k(G_1)$, $n_2 < \gamma_t^k(G_2)$.

Case (a): It is obvious that $n_2 = \gamma_t^k(G_2) + 1$. Suppose $G_1$ and $G_2$ are $\gamma_t^k$-critical graphs. Let $x \in N_k(u) \cap G_1$ and $y \in N_k(u) \cap G_2$. Let $S_x$, $S_y$ be the $\gamma_t^k$-set for $G_1 - x$ and $G_2 - y$. Let $S' = S_x \cup S_y \cup \{u, v\}$, for which $v \in N_u$. It is easy to check that $S'$ is a $\gamma_t^k$-set for $G$ with $\gamma_t^k(G_1) + \gamma_t^k(G_2)$ elements, a contradiction. Thus at most one component is $\gamma_t^k$-critical graph. Without lose of generality assume that $G_1$ is $\gamma_t^k$-critical graph. For $k = 1$, let $x \in G_1$ be a vertex with distance two from $u$.

We can total dominate $G$ with $S_x \cup S_2 \cup \{y\}$ where $y \in N(x) \cap N(u)$ and $S_x$ is total dominating set for $G_1 - x$, a contradiction. If $k \geq 2$ let $x \in N_k(u)$ and $y \in N_k(u) \cap N_k(x)$, we can total $k$-distance dominate $G$ with $S_x \cup S_2 \cup \{y\}$ where $S_x$ is $\gamma_t^k$-set for $G_1 - x$ with $\gamma_t^k(G_1) + \gamma_t^k(G_2)$ elements, a contradiction. So suppose both of $G_1$ and $G_2$ are not $\gamma_t^k$-critical graphs. Let $x \in G_2$. Since $G_2$ is not $\gamma_t^k$-critical graph, one of the following holds:

(a-1) $\gamma_t^k(G_2 - x) = \gamma_t^k(G_2)$;
(a-2) $\gamma_t^k(G_2 - x) > \gamma_t^k(G_2)$;
(a-3) $\gamma_t^k(G_2 - x) < \gamma_t^k(G_2)$.

In Case (a-1), if $\gamma_t^k(G_2 - x) = \gamma_t^k(G_2)$, then $\gamma_t^k(G - x) = \gamma_t^k(G_1) + \gamma_t^k(G_2)$, a contradiction.

In Case (a-2), if $\gamma_t^k(G_2 - x) > \gamma_t^k(G_2)$, then $\gamma_t^k(G - x) \geq \gamma_t^k(G_1) + \gamma_t^k(G_2) + 1$, a contradiction.

In Case (a-3), if $\gamma_t^k(G_2 - x) < \gamma_t^k(G_2)$, then $\gamma_t^k(G_2 - x) = \gamma_t^k(G_2) - 1$. Let $S_2$ be a $\gamma_t^k$-set for $G_2 - x$ and $y \in N_k(x) \cap N_k(u)$. It is easy to see that $S_x \cup S_1 \cup \{y\}$ is a $\gamma_t^k$-set for $G$ with $\gamma_t^k(G_1) + \gamma_t^k(G_2)$ elements, a contradiction.
Case (c): $n_1 < \gamma^k_t(G_1), n_2 > \gamma^k_t(G_2)$. If $G_2$ is $\gamma^k_t$-critical. Since $n_1 < \gamma^k_t(G_1)$, so some vertices of $G_1$ are $k$-distance dominated by elements of $G_2$. There is at least one vertex of $G_2$ which $k$-distance dominates some vertices of $G_1$. Let $s_1 \in S \cap G_2$ that $k$-distance dominate some vertices of $G_1$. Let $S_{s_1}$ be a $\gamma^k_t$-set for $G_2 - s_1$ with $\gamma^k_t(G_2) - 1$ elements and $S_1 := S \cap G_1$ with $n_1$ elements. It is clear that $S_{s_1} \cup S_1 \cup \{u, s_1\}$ is a $\gamma^k_t$-set for $G$ with $\gamma^k_t(G_2) + n_1 + 1$ elements which is less than $\gamma^k_t(G_1) + \gamma^k_t(G_2) + 1$, a contradiction. If $G_2$ is not $\gamma^k_t$-critical graph, then there is a vertex $x \in G_2$ such that $\gamma^k_t(G_2 - x) \geq \gamma^k_t(G_2)$. Therefore $\gamma^k_t(G - x) \geq \gamma^k_t(G)$, a contradiction.

Case (d): It is similar to Case (c). □

From Theorem 3.16 we have:

Corollary 3.17. If $T$ is a tree with size $n \geq 3$, then $T$ is not $\gamma^k_t$-critical graph.

4. On the equivalence of a conjecture

Mojdeh et al., in [7] conjectured: for $r \geq 6$, there is no 3-$\gamma_t$-critical $r$-regular graph of order $2r + 1$, that has been disproved by J. Rad et al., in [5]. Afterward it was also studied more by Sohn et al., in [9] and Wang et al, in [11]. Authors of [5] and [11] showed:

Theorem 4.1. ([5][11], Theorems 2.2, 2.1) For any even $r \geq 6$, if $M_r = \{2k - 1 : 1 \leq k \leq r/2\}$. The circulant graphs $G_r = C(2r + 1; M_r)$ are 3-$\gamma_t$-critical graphs of order $2r + 1$.

In this section we study a similar result on power $k$ of $G$, that is, we wish to study this problem when $\Delta_k(G) \geq 6$:

Theorem 4.2. For any even $k \geq 2$, there is a 3-$\gamma^k_t$-critical graph of order $3\frac{\Delta_t(G)}{2} + 2$.

Proof. Let $C_n$ be a cycle with size $n = 3k + 2$. Then $C^k_n$ is a Harary graph $G = H_{2k,n}$. Since $n = 3k + 2 = (3k + 1) + 1$, from Theorem 3.12 $H_{2k,n}$ is a 3-$\gamma_t$-critical graph of order $n = 3k + 2$. Therefore from Proposition 3.13 $C_n$ is a 3-$\gamma^k_t$-critical graph of order $3\frac{\Delta_t(G)}{2} + 2$. □

Also in [5] we have

Theorem 4.3. ([5], Theorem 2.1) A graph $G$ of order 9 is 3-$\gamma_t$-critical if and only if $G = F$, where $F$ be the graph with vertex set \{x, y, z, v_1, v_2, v_3, v_4, v_5, v_6\} and edge set \{xv_i : i = 1, 2, 3, 4\} ∪ \{v_1v_3, v_1v_5, v_1y, v_1v_6, v_2y, v_2v_6, v_3v_5, v_3z, v_4v_5, v_4z, v_4v_6, v_5y, yz, zv_6\}.

We can have corresponding result here.

Theorem 4.4. There is no 3-$\gamma^k_t$-critical graph with $\delta_k = \Delta_k = 4$ of order 9.
Proof. Let $G$ be a $3\-_{\gamma^k}$-critical graph of order 9 with $\delta_k = \Delta_k = 4$. Then $G^k$ should be a $3\-_{\gamma^l}$-critical 4-regular graph of order 9. Since with Theorem 4.3 the only $3\-_{\gamma^l}$-critical graph of order 9 is $F$, so $G^k = F$. In $F$ two vertices $z$ and $y$ are adjoint, therefore in $G$ we should have $N_k(y) \cap N_k(z) \neq \emptyset$, but in $F$ we have $N(y) \cap N(z) = \emptyset$, a contradiction. □

Now we can pose a conjecture as follows:

Conjecture. For $\Delta_k(G) \geq 6$, there is no $3\-_{\gamma^k}$-critical graph of order $2\Delta_k(G) + 1$.

we have followings from [8] and [9] respectively.

Theorem 4.5. ([8] Theorem 3.6) There is no $3\-_{\gamma^l}$-critical graph of order $\Delta(G) + 3$ with $\Delta(G) = 3, 5$ and $\delta(G) \geq 2$.

Theorem 4.6. ([8] Theorem 12) There is no $4\-_{\gamma^l}$-critical graph $G$ of order $\Delta(G) + 4$ with $\Delta(G) = 3, 5, 7$ and $\delta(G) \geq 2$.

Now we have corresponding results for power graph.

Theorem 4.7. (i) There is no $3\-_{\gamma^k}$-critical graph of order $\Delta_k(G) + 3$ with $\Delta_k(G) = 3, 5$ and $\delta_k(G) \geq 2$.

(ii) There is no $4\-_{\gamma^k}$-critical graph of order $\Delta_k(G) + 4$ with $\Delta_k(G) = 3, 5, 7$ and $\delta_k(G) \geq 2$.

Proof. (i) Suppose on the contrary, there is a graph $G$ which is $3\-_{\gamma^k}$-critical with the given properties. Then $G^k$ is a $3\-_{\gamma^l}$-critical graph of order $\Delta(G^k) + 3$ with $\Delta(G^k) = 3, 5$ and $\delta(G^k) \geq 2$, a contradiction.

(ii). This item has similar proof and it is left. □

To close of this paper, we pose the following problem:

Problem. How do we can generalize the properties of $\gamma^l$-criticality of a graph $G$ to the $\gamma^k$-criticality of $G$?

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